ODE

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Contents

1 Preamble

1.1 ODE forms

Definition 1.1.1

n-Order Odrdinary Differential Equation:

for $y = y(x)$, if we have

$$\begin{gathered} F\left(x, y, y', ..., y^{(n)}\right) = 0 \\ \text{or } y^{(n)}(x) = f\left(x, y, y', ..., y^{(n-1)}\right) \end{gathered} \tag{1}$$

then it's called **n-Order Odrdinary Differential Equation**

1.2 Solutions

Definition 1.2.1

n-Order ODE Solutions:

if $y = \varphi(x) \in C^n$ in I,

$$i.e.\ \ F(x, \varphi(x), \varphi'(x), \varphi^n(x)) \equiv 0,\ \forall x \in I \tag{2}$$

- $y = \varphi(x)$ is **a Particular solution** to the above ODE in I
- $y = \varphi(x, C_1, C_2, ..., C_n)$ is **General solutions** in I, if $C_1, C_2, ..., C_n$ satisfies:

$$\frac{D(\varphi, \varphi', \varphi'', ..., \varphi^{n-1})}{D(C_1, C_2, ..., C_n)} \stackrel{\text{def}}{=} \begin{vmatrix} \frac{\partial \varphi}{\partial C_1} & \frac{\partial \varphi}{\partial C_2} & \cdots & \frac{\partial \varphi}{\partial C_n} \\ \frac{\partial \varphi'}{\partial C_1} & \frac{\partial \varphi'}{\partial C_2} & \cdots & \frac{\partial \varphi'}{\partial C_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \varphi^n}{\partial C_1} & \frac{\partial \varphi^n}{\partial C_2} & \cdots & \frac{\partial \varphi^n}{\partial C_n} \end{vmatrix} \neq 0 \tag{3}$$

which means $C_1, C_2, ..., C_n$ are **Independent**

- if particular solution can not be expressed in form of general solution, then it's called **Singular solution**

Note

$I = [a, b],\ [a, b),\ (a, b],\ (a, b)$ is OK

$I = (1, 2) \cup (2, 3),\ (1, 2) \cup (3, 4)$ is wrong

1.3 Solution Methods

Note

- Guess
- Separation of Variables
- Integrating Factor
- Iteration
- Subsititution

1.4 Existence and Uniqueness of Solutions

- **Subsidiary Conditions:**
  - Boundary Conditions
  - Initial Conditions
- **隐函数定理**

Theorem 1.4.1

if $C_1, C_2, ..., C_n$ are Independent, for 定解问题（or 初值问题）

$$\begin{cases} y^{(n)} = f(x, y, y', ..., y^{(n-1)}) \\ (y, y', ..., y^{(n-1)})\big|_{x_0} = (y_0, y_1, ..., y_{n-1}) \end{cases} \tag{4}$$

if $\dfrac{D(\varphi, \varphi', ..., \varphi^{(n-1)})}{D(C_1, C_2, ..., C_n)} \neq 0$, in form of general solution we have

$$\begin{cases} C_1^* = C_1(x_0, y_0, y_1, ..., y_{n-1}) \\ C_2^* = C_2(x_0, y_0, y_1, ..., y_{n-1}) \\ \ldots\ldots \\ C_n^* = C_n(x_0, y_0, y_1, ..., y_{n-1}) \end{cases} \tag{5}$$

$\Longrightarrow$we get particular solution: $y = \varphi(x; C_1^*, C_2^*, ..., C_n^*)$

♡

1.5 Slope Fields and Solution Curves

Note

ODE $\Longleftrightarrow$ Slope Fields

Solutions $\Longleftrightarrow$ Intergral Curves

2 First-Order Differential Equation

2.1 Basic Concepts

Definition 2.1.1

1st-Order Odrdinary Differential Equation:

for $y = y(x)$, if we have

$$F(x, y, y') = 0 \text{ or } y' = f(x, y) \tag{6}$$

then it's called **1st-Order Odrdinary Differential Equation**

- Explicit

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{7}$$

- Implicit

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \tag{8}$$

Definition 2.1.2

if $y = \varphi(x) \in C^1(a, b)$ and satisfies

$$F(x, \varphi(x), \varphi'(x)) = 0 \text{ or } \varphi'(x) = f(x, \varphi(x)) \tag{9}$$

then $y = \varphi(x)$ is **a solution** to the above ODE on (a, b)

and for $y = \varphi(x, C_1)$ is **a set of solutions**

2.2 Linear First-Order ODE

Solution I - Integrating Factor

Forms

For **Linear 1st-Order ODE** like:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x) \\ y(x_0) = y_0 \end{cases} \tag{10}$$

$P(x), Q(x)$ are continuous.

Solutions

mutiply both sides by **Integrating factor** $e^{\int P(x)\,\mathrm{d}x}$, we have

$$\begin{aligned}\left(\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y\right)e^{\int P(x)\,\mathrm{d}x}&=Q(x)e^{\int P(x)\,\mathrm{d}x}\\ \Longrightarrow D_x\left(ye^{\int P(x)\,\mathrm{d}x}\right)&=Q(x)e^{\int P(x)\,\mathrm{d}x}\end{aligned}\tag{11}$$

- general solution:

$$y=e^{-\int P(x)\,\mathrm{d}x}\left(\int Q(x)e^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x+C_1\right)\tag{12}$$

- particular solution:

$$y=e^{-\int_{x_0}^{x}P(s)\,\mathrm{d}s}\left(\int_{x_0}^{x}Q(x)e^{\int_{x_0}^{s}P(t)\,\mathrm{d}t}\,\mathrm{d}s+y_0\right)\tag{13}$$

2.2.1 Homogeneous LODE

$$y'=P(x)y\tag{14}$$

1. $y\equiv 0$ or $y\not\equiv 0$, $\forall x\in I$

Proof by contradiction:

$$\text{Assume } \exists y=\varphi(x), s.t.\ \varphi(x_1)=0 \text{ and } \forall x\in(x_1,x_2], \varphi(x)>0\tag{15}$$

$$\Rightarrow \forall x\in[x_3,x_2], \varphi'(x)=P(x)\varphi(x)\tag{16}$$

$$\Rightarrow \int_{x_3}^{x_2}\varphi'/\varphi\,\mathrm{d}s=P(x)\tag{17}$$

$$\Rightarrow \lim_{x_3\to x_1}\int_{x_3}^{x_2}\varphi'/\varphi\,\mathrm{d}s=\int_0^{\varphi(x_2)}\mathrm{d}\varphi/\varphi\to\infty\tag{18}$$

Contradiction.

1. $P(x)\in C(I)$ $\Rightarrow$解在I上存在
2. 其解集组成一维线性空间

2.2.2 Non-homogeneous LODE

$$y'=P(x)y+Q(x)\tag{19}$$

1. 初值问题下存在唯一解

$$\begin{cases}y'=P(x)y+Q(x)\\ y(x_0)=y_0\end{cases}\tag{20}$$

(Uniquness) Proof by contradiction:

$$\begin{aligned} &\text{Assume } \exists \text{ Solution } y_1 \neq y_2 \\ &\Rightarrow (y_1 - y_2)' = P(x)(y_1 - y_2) \end{aligned} \tag{21}$$

- $y_1 - y_2 \equiv 0$, Contradiction.
- $y_1 - y_2 \not\equiv 0$ but $(y_1 - y_2)(x_0) = 0$, Contradiction.

1. $P(x), Q(x) \in C(I) \Rightarrow$解在$I$上存在
2. 通解结构：Non-homogeneous 通解 = Homogeneous 通解 + Non-homogeneous 特解
3. 叠加原理：$c_1y_1 + c_2y_2 \begin{cases} y_1' = P(x)y_1' + Q_1(x) \\ y_2' = P(x)y_2' + Q_2(x) \end{cases}$ 为 $y' = P(x)y + c_1Q_1(x) + c_2Q_2(x)$的解

2.3 Non-linear First-Order ODE

Solution II - Separable Equations

Forms

For **1st-Order ODE** like:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = H(x,y) = g(x)g(y)y|_{x=x_0} = y_0 \end{cases} \tag{22}$$

Solutions

- $h(y) \neq 0$

$$\int \frac{\mathrm{d}y}{h(y)} = \int g(x)\,\mathrm{d}x \Longrightarrow y = \varphi(x, C_1) \tag{23}$$

- $h(y_0) = 0$

$$y \equiv y_0 \text{ is a constant solution} \tag{24}$$

Example 1.

$$y' = \frac{y}{2x - y^2} \tag{25}$$

but for x, we have

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{2x - y^2}{y} = \frac{2}{y}x - y \\ &\Longrightarrow x = y^2(C - \ln|y|) \text{ or } y \equiv 0 \end{aligned} \tag{26}$$

Example 2: Logistic Model(人口模型).

$$\begin{cases}\frac{\mathrm{d}P}{\mathrm{d}x}=kP\left(1-\frac{P}{M}\right)P(x=0)=P_0\end{cases} \tag{27}$$

Step 1. if $P_0\neq 0, P_0\neq M$

$$\int\frac{\mathrm{d}P}{P(M-P)}=\int Mk\,\mathrm{d}x$$

$$\ln\left|\frac{P}{M-P}\right|=Mkx \tag{28}$$

$$\frac{M-P}{P}=\pm Ce^{-Mkx},\ P=\frac{P_0M}{P_0+(M-P_0)e^{-Mkx}}$$

Step 2. if $P_0=0$ or $P_0=M$

$$P=\frac{P_0M}{P_0+(M-P_0)e^{-Mkx}} \tag{29}$$

still holds true. □

2.3.1 Separable Equations

- 初值问题下不一定存在唯一解

Example 1.

$$\begin{cases}y'=\frac{3}{2}y^{1/3}\\ y(0)=0\end{cases} \tag{30}$$

Step 1. $y\equiv 0$

Step 2. $y\neq 0,\ \frac{2}{3}\frac{\mathrm{d}y}{y^{1/3}}=\mathrm{d}x\Rightarrow y=\pm(x+C)^{3/2}$

But for $y=\begin{cases}0,\ x\in[0,1]\\(x-1)^{3/2},\ x>1\end{cases}$ still satisfies. $\Longrightarrow\infty$ solutions

Form

$$\begin{cases}y'=P(x)y^{\alpha},\ \alpha>0\\ y(0)=0\end{cases} \tag{31}$$

- $\alpha=1,\exists!$ solution
- $\alpha<1,\infty$ solutions
- $\alpha>1,\exists!$ solution

- 系数连续解不一定在该区间上连续
- 若存在解的区间，不同初值会导致解的存在区间不同

- y有界，y'不一定有界

Solution III - Subsititution

Forms

For **Homogeneous 1st-Order ODE** like:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F\left(\frac{y}{x}\right) \tag{32}$$

Solutions

let $v = \frac{y}{x}$, then $y = vx$, we have

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = F(v) - v \tag{33}$$

Step 1. $F(v) - v \neq 0$, then it's a separable equation

$$\begin{gathered}\frac{\mathrm{d}v}{F(v) - v} = \frac{\mathrm{d}x}{x} \\ \Longrightarrow F(v) - v = Cx \Longrightarrow F\left(\frac{y}{x}\right) - \frac{y}{x} = Cx\end{gathered} \tag{34}$$

Step 2. $F(v_0) - v_0 = 0$, then $\frac{y}{x} = v_0 \Longrightarrow y = v_0 x$ is a constant solution

2.3.2 Homogeneous Equations

Definition 2.3.1

n-th order Homogeneous Function:

for $F(x, y)$, if $\forall\lambda \neq 0$, $F(\lambda x, \lambda y) = \lambda^n F(x, y)$, then it's called **n-th order Homogeneous Function**

Example 1.

$$y' = \frac{x + y}{x - y} \tag{35}$$

Step 1. $\forall\lambda \neq 0, f(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \frac{x + y}{x - y}$ $\Longrightarrow$1st order Homogeneous Equations

Step 2. let$\lambda = \frac{1}{x}$, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + y/x}{1 - y/x} \Longrightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = F(v) - v$

Example 2.

$$y' = \frac{x + y + 1}{x - y + 2} \tag{36}$$

find $\begin{cases}\xi = x+\alpha \\ \gamma = y+\beta\end{cases}$, then we have

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\xi} = \frac{\xi+\gamma}{\xi-\gamma} \tag{37}$$

and

$$\begin{cases}\alpha = \dfrac{3}{2} \\ \beta = -\dfrac{1}{2}\end{cases} \tag{38}$$

Example 3.

$$y' = x^{k-1} f\left(\frac{y}{x^k}\right) \tag{39}$$

let $u = y/x^k$, then $u' = \dfrac{1}{x}(f(u) - u)$

Example 4.

$$xy' = f(xe^{-y}) \tag{40}$$

let $u = xe^{-y}$, then $y = \ln x - \ln u$, $xu' = u(1 - f(u))$

Example 5.

$$y' = yf(ye^{-x}) \tag{41}$$

- let $u = ye^{-x}$, then $y = ue^x$, $u' = u(f(u) - 1)$
- let $t = e^x$, then $x = \ln t$, $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{y}{t} f\left(\dfrac{y}{t}\right)$

Example 6.

$$y' = \frac{y}{x} + h(x) f\left(\frac{y}{x}\right) \tag{42}$$

$$xy' = y + h(x) f\left(\frac{y}{x}\right) \tag{43}$$

$$y' = \frac{y}{x + f\left(\frac{y}{x}\right)} \tag{44}$$

$$xy' = \frac{y}{\ln x + f(y)} \tag{45}$$

2.3.3 Bernoulli's Equations (Jakob, 1695)

Forms

For **Bernoulli's Equations** like:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^\alpha,\ \alpha \neq 0,1 y(x_0) = y_0 \end{cases} \tag{46}$$

$P(x), Q(x)$ are continuous.

Solutions

let $z = y^{1-\alpha}$, then $y = z^{1/(1-\alpha)}$, we have

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-\alpha)P(x)z = (1-\alpha)Q(x) \tag{47}$$

which is a linear equation in z.

if $\alpha > 0$, then $y \equiv 0$ is a constant solution.

2.3.4 Ricatti's Equations

Forms

For **Riccati's Equations** like:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y + Q(x)y^2 = K(x) y(x_0) = y_0 \end{cases} \tag{48}$$

$P(x), Q(x), K(x)$ are continuous.

Solutions

if we have a particular solution y_1, let $u = y - y_1$, then u satisfies

$$\frac{\mathrm{d}u}{\mathrm{d}x} + P(x)u + 2Q(x)y_1u + Q(x)u^2 = 0 \tag{49}$$

which is a **Bernoulli's Equations** in u.

Special Forms(D.Bernoulli, 1725; Liouville, 1841)

$$y' + y^2 = bx^m,\ m = 0, -2, -\frac{4k}{2k \pm 1} \tag{50}$$

Example 1.

$$y' + y^2 = \frac{6}{x^2} \tag{51}$$

- Guess $y_1 = \frac{c}{x} \Longrightarrow c = -2, 3$, let $u = y - y_1$ we have

$$u' + \frac{6}{x}u + u^2 = 0 \tag{52}$$

- let $u = xy \Longleftrightarrow y = u/x$, we have

$$u' = \frac{6 - u^2 + u}{x} \tag{53}$$

Example 2.

$$y' + y^2 = bx^m \tag{54}$$

find (ξ, η) *s.t.*

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} + \eta^2 = b\xi^n \tag{55}$$

$$\alpha : \begin{cases} y = \frac{1}{p\eta} \\ x = \left(\frac{m+1}{pb}\xi\right)^{1/(m+1)} \end{cases} \Longrightarrow \frac{\mathrm{d}\eta}{\mathrm{d}\xi} + \eta^2 = \frac{1}{p^2 b}\left(\frac{m+1}{pb}\xi\right)^{-m/(m+1)}, n = -\frac{m}{m+1}$$

$$\beta : \begin{cases} y = \xi - \xi^2\eta \\ x = \frac{1}{\xi} \end{cases} \Longrightarrow \frac{\mathrm{d}\eta}{\mathrm{d}\xi} + \eta^2 = b\xi^{-m-4}, n = -m-4$$

Key: $\begin{cases} y = F(\eta) \\ x = G(\xi) \end{cases}$

2.3.5 Total Derivative Equations

Forms

For **Total Derivative Equations** like:

$$\begin{cases} M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = \mathrm{d}F(x,y) = 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{x=x_0} = y_0 \end{cases} \tag{56}$$

$M(x,y), N(x,y) \in C'(D), N(x,y) \neq 0$. D is **Simply-Connected**

Solutions

we set $\boldsymbol{F} = (M, N)$, if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ or } \nabla \times \boldsymbol{F} = 0 \tag{57}$$

, then it's **an Exact Equation**, and we have

$$\exists F(x,y), s.t. \ \frac{\partial F}{\partial x} = M(x,y),\ \frac{\partial F}{\partial y} = N(x,y) \tag{58}$$

$$\Longrightarrow F(x,y) = C_1 \ \text{(General Solution)} \tag{59}$$

if

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ or } \nabla \times \boldsymbol{F} \neq 0 \tag{60}$$

, let $\mu(x,y) \neq 0$ be an **Integrating Factor**, s.t.

$$\mu M \,\mathrm{d}x + \mu N \,\mathrm{d}y = \mathrm{d}G = 0 \tag{61}$$

is an Exact Equation.

$$\mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = N\frac{\partial \mu}{\partial y} - M\frac{\partial \mu}{\partial x} \tag{62}$$

- **2-Dimension:**

$$\begin{gathered}
\nabla \times (\mu \boldsymbol{F}) = 0 \\
\Longrightarrow \nabla\mu \times \boldsymbol{F} + \mu\nabla \times \boldsymbol{F} = 0 \\
\Longrightarrow \boldsymbol{F} \times (\nabla\mu \times \boldsymbol{F} + \mu\nabla \times \boldsymbol{F}) = 0 \\
\Longrightarrow \nabla \ln\mu = \frac{\nabla \ln\mu \cdot \boldsymbol{F}}{\|\boldsymbol{F}\|^2}\boldsymbol{F} - \frac{\boldsymbol{F} \times (\nabla \times \boldsymbol{F})}{\|\boldsymbol{F}\|^2} \\
\Longrightarrow \begin{cases} \partial_x \ln\mu = \dfrac{M}{N}\partial_y \ln\mu - \dfrac{\|\nabla \times \boldsymbol{F}\|}{N} \\ \partial_y \ln\mu = \dfrac{N}{M}\partial_x \ln\mu + \dfrac{\|\nabla \times \boldsymbol{F}\|}{M} \end{cases} \Longrightarrow \begin{cases} \mu = \mu(x) \Rightarrow \mu = e^{-\int (\|\nabla \times \boldsymbol{F}\|)/N} \\ \mu = \mu(y) \Rightarrow \mu = e^{\int (\|\nabla \times \boldsymbol{F}\|)/M} \\ \mu = \mu(x,y) = \begin{cases} \text{General: } \mu_i(x,y)g_i(G_i), \begin{cases} \mathrm{d}G = \sum_i \mathrm{d}G_i = \sum_i \mu_i \,\mathrm{d}F_i \\ F = \sum_i \mathrm{d}F_i \\ \mathrm{d}G \equiv \mu_i g_i(G_i) F \end{cases} \\ \text{Homogeneous: } \mu(x,y) = 1/(xM + yN) \end{cases} \end{cases}
\end{gathered} \tag{63}$$

- **3-Dimension:**

$$\begin{gathered}
\nabla \times (\mu \boldsymbol{F}) = 0 \\
\Longrightarrow \nabla\mu \times \boldsymbol{F} + \mu\nabla \times \boldsymbol{F} = 0 \\
\Longrightarrow \nabla\mu \cdot (\nabla\mu \times \boldsymbol{F} + \mu\nabla \times \boldsymbol{F}) = 0 \\
\Longrightarrow \nabla\mu \cdot (\nabla \times \boldsymbol{F}) = 0
\end{gathered} \tag{64}$$

♡

	Nature of Problem	Integral Factor	After Correction
Field Theorem	$\boldsymbol{F} = (M, N), \nabla \times \boldsymbol{F} \neq 0$	Scalar Field	$\boldsymbol{G} = (\mu M, \mu N), \nabla \times \boldsymbol{G} = 0, \boldsymbol{G} = \nabla g$
Differential Form	1-form: $\omega = M\,\mathrm{d}x + N\,\mathrm{d}y$ is not closed	0-form	new 1-form: $\eta = \mu\omega$ is closed and $\eta = \mathrm{d}g$
Coordinate Transformation	(x, y) is twisted	A Transformation: $(x, y) \to (X, Y) \to (0, G)$	$(0, G)$ is straight (Trivial!) also for $\mu = \mu(x + y), \mu(xy), \mu(F)$

2.3.6 Implicit Form and Singular Solution

Forms

$$y' = f(x, y) \Longrightarrow F(x, y, y') = 0 \tag{65}$$

On Space of (x, y, y') we have 2-D **Surface** $F(x, y, y') = 0$, then projection to planar x-y we have **Solution Curve**

Solution

Step 1. For $F(x, y, p) = 0$ we have parameter equation

$$\{x = x(s, t)y = y(s, t)p = p(s, t) \tag{66}$$

Step 2. $\mathrm{d}y = p\,\mathrm{d}x$

$$\Longrightarrow \mathrm{d}y(s, t) = y_s\,\mathrm{d}s + y_t\,\mathrm{d}t = p(s, t)(x_s\,\mathrm{d}s + x_t\,\mathrm{d}t) \tag{67}$$

Step 3. Eq.(67) to Eq.(66)

Definition 2.3.2

For $(*)$ $F(x, y, y') = 0$, if $y = \varphi(x)$ is a solution on I we have solution curve $\Gamma = \{(x, \varphi(x)), x \in I\}$ if $\forall p \in \Gamma$, $\exists$ Another solution (partialerent in Γ) is Tangent with p in $U(p)$, we call it as **Singular Solution**

- **Necessary Condition:**

$$\begin{cases} F(x, y, p) = 0 \\ F_p(x, y, p) = 0, \ \mathrm{d}y = p\,\mathrm{d}x \end{cases} \tag{68}$$

- **Sufficient Condition:**

$$\begin{cases} F(x, y, p) = 0 \\ F_p(x, y, p) = 0, \ \mathrm{d}y = p\,\mathrm{d}x \\ F_y(x, y, p) \neq 0 \\ F_{pp}(x, y, p) \neq 0 \end{cases} \tag{69}$$

> Assume $F, F_y, F_p \in C(I)$

2.3.7 Order-Reduction of Higher Order ODEs

Forms

For $y'' = f(x, y, y')$

if $y'' = f(x) \Longrightarrow$just intergral

if $y'' = f(x, y') \Longrightarrow$let $p = y', p' = f(x, p)$

———— if $y^{(n)} = f\big(x, y^{(n-1)}\big) \Longrightarrow$let $p = y^{(n-1)}, p' = f(x, p)$

if $y'' = f(y, y') \Longrightarrow$let $p = y', \dfrac{\mathrm{d}p}{\mathrm{d}x} = p\dfrac{\mathrm{d}p}{\mathrm{d}y} = f(y, p)$

Example 1. For

$$k = \frac{|y''|}{(1 + y'^2)^{3/2}} = \text{const} \tag{70}$$

let $p = y'$, then

$$\begin{aligned} k = \frac{|p'|}{(1+p^2)^{3/2}} &\Longrightarrow \frac{p}{\sqrt{1+p^2}} = k(x + C_1) \\ \Longrightarrow y' = p = \frac{k(x+C_1)}{\sqrt{1 - k^2(x+C_1)^2}} &= \frac{x + C_1}{\sqrt{\dfrac{1}{k^2} - (x+C_1)^2}} \end{aligned} \tag{71}$$

$$\Longrightarrow y(x) = -\sqrt{\frac{1}{k^2} - (x+C_1)^2} + C_2 \tag{72}$$

$$\Longrightarrow (y - C_2)^2 + (x + C_1)^2 = \frac{1}{k^2} \tag{73}$$

Example 2.

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \tag{74}$$

let $p = \dfrac{\mathrm{d}y}{\mathrm{d}x}$, then

$$p^2 - yp\frac{\mathrm{d}p}{\mathrm{d}y} \tag{75}$$

$$\Longrightarrow p = 0 \text{ or } p - y\frac{\mathrm{d}p}{\mathrm{d}y} = 0 \tag{76}$$

$$\Longrightarrow y = C_2 e^{C_1 x} \tag{77}$$

3 LODEs and Higher Order LODEs

3.1 Formalization of LODEs

Definition 3.1.1

LODE Standard Form:

$$y^{(n)} = f(t) + p_0(t)y + p_1(t)y' + \dots + p_{n-1}(t)y^{(n-1)} \tag{78}$$

let

$$\begin{cases} x_1(t) = y(t) \\ x_2(t) = y'(t) \\ \dots \\ x_n(t) = y^{(n-1)}(t) \end{cases} \tag{79}$$

then

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_1 & p_2 & p_3 & \dots & p_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dots \\ f(t) \end{pmatrix} \tag{80}$$

or

$$\boldsymbol{X}' = \boldsymbol{A}(t)\boldsymbol{X} + \boldsymbol{B}(t) \tag{81}$$

♣

Forms(Homogeneous)

For $A, B \in C(I)$

$$\boldsymbol{X}'(t) = \boldsymbol{A}(t)\boldsymbol{X}(t) \tag{82}$$

General Solution:

$$\boldsymbol{X}(t) = (\boldsymbol{X}_1(t), \boldsymbol{X}_2(t), \dots, \boldsymbol{X}_n(t)) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \overset{\text{def}}{=} \boldsymbol{\Phi}(t)\boldsymbol{C} \tag{83}$$

Lemma 3.1.2 (Wronsky Determinant)

$(\boldsymbol{X}_1(t), \boldsymbol{X}_2(t), \dots, \boldsymbol{X}_n(t))$ are independent solutions $\Leftrightarrow \det(\boldsymbol{\Phi}(t)) = W(t) \neq 0, \forall t \in I$

♡

Lemma 3.1.3 (Liouville Formula)

$(\boldsymbol{X}_1(t), \boldsymbol{X}_2(t), \dots, \boldsymbol{X}_n(t))$ are solutions $\Rightarrow W(t) = W(t_0)e^{\int_{t_0}^{t} \mathrm{Tr}(\boldsymbol{A}(s))\,\mathrm{d}s}$

♡

♠

Theorem 3.1.4 (Existence and Uniqueness)

For (*)

$$\begin{cases} \boldsymbol{X}'(t) = \boldsymbol{A}(t)\boldsymbol{X}(t) + \boldsymbol{B}(t) \\ \boldsymbol{X}(t_0) = \boldsymbol{X}_0 \end{cases} \tag{84}$$

and $A, B \in C(I), t_0 \in I = [a, b] \implies \exists!$ Solution $\boldsymbol{X}(t)$ in I

Existence Proof.

Proof. I. (*)$\iff \boldsymbol{X} = \boldsymbol{X}_0 + \int_{t_0}^t [A\boldsymbol{X} + B]\,\mathrm{d}s$

II. **Picard Iteration Series**

$$\boldsymbol{X}_0(t) \equiv \boldsymbol{X}_0 \tag{85}$$

$$\boldsymbol{X}_{k+1}(t) = \boldsymbol{X}_0 + \int_{t_0}^t [\boldsymbol{A}(s)\boldsymbol{X}_k(s) + \boldsymbol{B}(s)]\,\mathrm{d}s \tag{86}$$

III. **Cauchy Series**

let $t \in [t_0, b] = J$

$$\begin{aligned} \|\boldsymbol{X}_1(t) - \boldsymbol{X}_0(t)\| &= \left\| \int_{t_0}^t [A\boldsymbol{X}_0 + B]\,\mathrm{d}s \right\| \\ &\le \int_{t_0}^t [\|A\boldsymbol{X}_0\| + \|B\|]\,\mathrm{d}s \le \int_{t_0}^t [\|A\|\|\boldsymbol{X}_0\| + \|B\|]\,\mathrm{d}s \\ &\le \left\{ \sup_{t\in I}\|A\|\|\boldsymbol{X}_0\| + \sup_{t\in I}\|B\| \right\}(t - t_0) \end{aligned} \tag{87}$$

let $C = \sup\|A\|, D = C\|\boldsymbol{X}_0\| + \sup\|B\|$

由归纳法

$$\begin{aligned} \|\boldsymbol{X}_k - \boldsymbol{X}_{k-1}\| &\le \frac{D}{C}\frac{[C(t-t_0)]^k}{k!} \\ \|\boldsymbol{X}_{k+1} - \boldsymbol{X}_k\| &= \left\| \int_{t_0}^t A(\boldsymbol{X}_k - \boldsymbol{X}_{k-1})\,\mathrm{d}s \right\| \le \left\| \int_{t_0}^t C\|\boldsymbol{X}_k - \boldsymbol{X}_{k-1}\|\,\mathrm{d}s \right\| \\ &\le \int_{t_0}^t C\frac{D}{C}\frac{[C(t-t_0)]^k}{k!}\,\mathrm{d}s = \frac{DC^k(t-t_0)^{k+1}}{(k+1)!} \end{aligned} \tag{88}$$

$$\begin{aligned} \|\boldsymbol{X}_m - \boldsymbol{X}_k\| &\le \sum_{j=k+1}^m \|\boldsymbol{X}_k - \boldsymbol{X}_{k-1}\| \le \frac{D}{C}\sum_{k+1}^m \frac{[C(t-t_0)]^k}{k!} \\ &\le \frac{D}{C}\sum_{k+1}^m \frac{[C(b-a)]^k}{k!} \to 0 \text{ which is Taylor extension of } e^{b-a} \text{ at } (b-a) \to 0 \\ &\implies \exists \varepsilon > 0, \forall m, k > 0, \forall t \in J \ \text{ s.t. } \|\boldsymbol{X}_m - \boldsymbol{X}_k\| < \varepsilon \quad \text{一致收敛} \end{aligned} \tag{89}$$

□

thus

$$\begin{aligned}\lim_{n\to\infty} \boldsymbol{X}_n(t) = \boldsymbol{X}(t) &= \boldsymbol{X}_0 + \lim_{n\to\infty} \int_{t_0}^{t} [\boldsymbol{A}(s)\boldsymbol{X}_{n-1}(s) + \boldsymbol{B}(s)]\,\mathrm{d}s \\ &= \boldsymbol{X}_0 + \int_{t_0}^{t} \lim_{n\to\infty} [\boldsymbol{A}(s)\boldsymbol{X}_{n-1}(s) + \boldsymbol{B}(s)]\,\mathrm{d}s \\ &= \boldsymbol{X}_0 + \int_{t_0}^{t} [\boldsymbol{A}(s)\boldsymbol{X}(s) + \boldsymbol{B}(s)]\,\mathrm{d}s\end{aligned} \tag{90}$$

Uniqueness Proof.

Proof. If $\exists \boldsymbol{X}, \boldsymbol{Y}$, then

$$\begin{aligned}\boldsymbol{X}(t) - \boldsymbol{Y}(t) &= \int_{t_0}^{t} \boldsymbol{A}(s)(\boldsymbol{X}(s) - \boldsymbol{Y}(s))\,\mathrm{d}s \\ \Longrightarrow \|\boldsymbol{X}(t) - \boldsymbol{Y}(t)\| &\le \int_{t_0}^{t} \|\boldsymbol{A}(s)\|\|\boldsymbol{X}(s) - \boldsymbol{Y}(s)\|\,\mathrm{d}s \\ \Longrightarrow \|\boldsymbol{X} - \boldsymbol{Y}\|(t) &\le 0 \cdot \exp\left(\int_{t_0}^{t} \|\boldsymbol{A}(s)\|\|\boldsymbol{X}(s) - \boldsymbol{Y}(s)\|\,\mathrm{d}s\right) = 0 \text{ (Gronwell)} \\ &\Longrightarrow \boldsymbol{X} = \boldsymbol{Y}\end{aligned} \tag{91}$$

□

Forms(Inhomogeneous)

For $A, B \in C(I)$

$$\begin{cases}\boldsymbol{X}'(t) = \boldsymbol{A}(t)\boldsymbol{X}(t) + \boldsymbol{B}(t) \\ \boldsymbol{X}(t_0) = \boldsymbol{X}_0\end{cases} \tag{92}$$

Special Solution:

$$\boldsymbol{X}^*(t) = \boldsymbol{\Phi}(t)\boldsymbol{C}(t) = \boldsymbol{\Phi}(t) \int_{t_0}^{t} \boldsymbol{\Phi}^{-1}(s)\boldsymbol{B}(s)\,\mathrm{d}s \tag{93}$$

General Solution:

$$\begin{aligned}\boldsymbol{X}(t) &= \boldsymbol{X}^*(t) + \boldsymbol{\Phi}(t)\boldsymbol{C} \\ &= \boldsymbol{\Phi}(t)\left(\int_{t_0}^{t} \boldsymbol{\Phi}^{-1}(s)\boldsymbol{B}(s)\,\mathrm{d}s + \boldsymbol{C}\right) \\ &= \boldsymbol{\Phi}(t)\int_{t_0}^{t} \boldsymbol{\Phi}^{-1}(s)\boldsymbol{B}(s)\,\mathrm{d}s + \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(t_0)\boldsymbol{X}_0\end{aligned} \tag{94}$$

3.2 Solution of LODEs

Theorem 3.2.1 (Homogeneous)

For $A, B \in C(I)$

$$\begin{cases} \boldsymbol{X}'(t) = \boldsymbol{A}(t)\boldsymbol{X}(t) \\ \boldsymbol{X}(t_0) = \boldsymbol{X}_0 \end{cases} \tag{95}$$

if **Exchangeability** satisfies

$$\boldsymbol{A}(t) \cdot \int_{t_0}^{t} \boldsymbol{A}(s)\,\mathrm{d}s = \int_{t_0}^{t} \boldsymbol{A}(s)\,\mathrm{d}s \cdot \boldsymbol{A}(t) \tag{96}$$

then

$$\boldsymbol{X}(t) = e^{\boldsymbol{D}(t)}\boldsymbol{X}_0 = e^{\int_{t_0}^{t}\boldsymbol{A}(s)\,\mathrm{d}s}\boldsymbol{X}_0 = \sum_{m=0}^{\infty}\frac{\boldsymbol{D}^m(t)}{m!}\boldsymbol{X}_0 \tag{97}$$

if $\boldsymbol{A}$ is constant matrix

- $\boldsymbol{A}$ has n independent eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n) = \boldsymbol{\Lambda} \Longrightarrow \boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{\Lambda}, \boldsymbol{P} = (\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n) \tag{98}$$

$$\Longrightarrow \boldsymbol{X}_i(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{v}_i = e^{\lambda_i(t-t_0)}\boldsymbol{v}_i,\ \boldsymbol{v}_i = \boldsymbol{X}_i(t_0) \tag{99}$$

$$\begin{gathered} \text{Assume } t_0 = 0 \Longrightarrow \boldsymbol{X}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}\boldsymbol{v}_i = \boldsymbol{P}e^{\boldsymbol{\Lambda}t}\boldsymbol{C} = e^{\boldsymbol{A}t}\underbrace{\boldsymbol{P}\boldsymbol{C}}_{\boldsymbol{X}_0} = \boldsymbol{\Phi}(t)\boldsymbol{C} \\ \boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t}\boldsymbol{P},\ \boldsymbol{\Phi}(0) = \boldsymbol{P},\ e^{\boldsymbol{A}t} = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(0) \end{gathered} \tag{100}$$

Note

if $\lambda_{r+1}, \lambda_{r+2}, ..., \lambda_n$ are complex,we can have real value solutions:

$$\begin{gathered} \boldsymbol{\Phi}(t) = \left(e^{\lambda_1 t}\boldsymbol{v}_1, e^{\lambda_2 t}\boldsymbol{v}_2, ..., e^{\lambda_r t}\boldsymbol{v}_r, \mathrm{Re}(e^{\lambda_{r+1}t}\boldsymbol{v}_{r+1}), \mathrm{Im}(e^{\lambda_{r+1}t}\boldsymbol{v}_{r+1}), ..., \mathrm{Im}(e^{\lambda_{r+s}t}\boldsymbol{v}_{r+s})\right) \\ s = (n-r)/2 \end{gathered} \tag{101}$$

- $\boldsymbol{A}$ have k(k < n) independent eigenvectors, each multiplicity number is $n_j, 1 \le j \le k$

$$\left(\boldsymbol{A} - \lambda_j\boldsymbol{E}\right)^{\beta}\boldsymbol{v}_j^l = 0\ \left(\beta \ge n_j\right),\ \boldsymbol{A} = \boldsymbol{P}\boldsymbol{J}\boldsymbol{P}^{-1} \tag{102}$$

$$\begin{gathered} \boldsymbol{X}_j^l(t) = e^{\boldsymbol{A}t}\boldsymbol{v}_j^l = e^{\lambda_j t}e^{(\boldsymbol{A}-\lambda_j\boldsymbol{E})t}\boldsymbol{v}_j^l \\ = e^{\lambda_j t}\left[\boldsymbol{E} + t(\boldsymbol{A} - \lambda_j\boldsymbol{E}) + \frac{t^2}{2!}\left(\boldsymbol{A} - \lambda_j\boldsymbol{E}\right)^2 + ... + \frac{t^{n_j-1}}{(n_j-1)!}\left(\boldsymbol{A} - \lambda_j\boldsymbol{E}\right)^{n_j-1}\right]\boldsymbol{v}_j^l = e^{\boldsymbol{J}_j t}\boldsymbol{v}_j^l \end{gathered} \tag{103}$$

$$\boldsymbol{\Phi}(t) = \left(\boldsymbol{X}_1^1, \boldsymbol{X}_1^2, ..., \boldsymbol{X}_1^{n_1}, ..., \boldsymbol{X}_k^{n_k}\right)(t) \tag{104}$$

♡

4 General Theory of ODE

4.1 Existence and Uniqueness of Initial Value Problems

Definition 4.1.1 (Equicontinuous)

For $\{f_n(x)\}$ on $I = [a, b]$

if $\forall \varepsilon > 0$, $\forall n = 1, 2, ..., \exists \delta = \delta(\varepsilon) > 0$, s.t. $\forall |x_i - x_j| < \delta \Rightarrow |f_n(x_i) - f_n(x_j)| < \varepsilon$

then $\{f_n(x)\}$ is **Equicontinuous** on I.

Definition 4.1.2 (Uniformly Bounded)

For $\{f_n(x)\}$ on $I = [a, b]$

if $\exists K$, s.t. $\forall n = 1, 2, ...$, $\forall x \in I$, $|f_n(x)| \leq K$

then $\{f_n(x)\}$ is **Uniformly Bounded** on I.

Lemma 4.1.3

$A \subset I$ is **Dense Point Set** of I, $\{f_n\{x\}\}$ is **Equicontinuous** over I,

if $\forall x_i \in A$, $\{f_n\{x_i\}\}$ is **Cauchy Sequence**, then $\{f_n\{x\}\} \rightrightarrows f(x)$ is Continuous on I.

Theorem 4.1.4 (Arzela-Ascoli Theorem)

For $\{f_n(x)\}$ on $I = [a, b]$

if $\{f_n(x)\}$ is **Equicontinuous** and **Uniformly Bounded** on I,

then at least $\exists$ one $\left\{f_{n_k}(x)\right\}$, s.t. $f_{n_k}(x) \rightrightarrows f(x)$ is Contineuous on I

4.1.1 Peano Theorem of Existence

Theorem 4.1.5 (Peano Theorem)

For

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \\ y(x_0) = y_0 \end{cases} \tag{105}$$

if $f(x, y) \in C(D)$, $D = \{(x, y) |\ |x - x_0| \leq a, |y - y_0| \leq b\}$

then at least $\exists$ one $y = \varphi(x)$ on $[x_0 - h, x_0 + h]$,

$$h = \min\left(a, \frac{b}{M}\right), \quad M = \sup_{\{(x,y)\in D\}} |f(x, y)| \tag{106}$$

- 找到 Euler 折线⇒证明近似解⇒证明满足 Arzela-Ascoli 定理⇒证明存在解

4.1.2 Picard Theorem of Uniqueness and Existence

Definition 4.1.6 (Lipschitz Condition)

For $f(x,y)$ on G

if for $\forall$ Bounded Close Set $K \subset G$, $\exists L_K > 0$, s.t.

$$\forall (x,y_1),(x,y_2) \in G,\ |f(x,y_1) - f(x,y_2)| \leq L_K |y_1 - y_2| \tag{107}$$

then $f(x,y)$ satisfies **Lipschitz Condition** on G.

if L_K is independent of K, then $f(x,y)$ satisfies **Global Lipschitz Condition** on G.

Theorem 4.1.7 (Picard Theorem)

For

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \\ y(x_0) = y_0 \end{cases} \tag{108}$$

if $f(x,y), \dfrac{\partial f}{\partial y} \in C(D)$, $D = \{(x,y) |\ |x - x_0| \leq a, |y - y_0| \leq b\}$

then $\exists!$ one $y = \varphi(x)$ on $[x_0 - h, x_0 + h]$,

$$h = \min\left(a, \frac{b}{M}\right), \quad M = \sup_{\{(x,y)\in D\}} |f(x,y)| \tag{109}$$

- 存在性：找到 Picard 迭代序列⇒证明连续可微+一致有界⇒证明一致收敛⇒证明存在连续解
- 唯一性：假设存在两个解⇒构造差值函数⇒利用 Gronwall 不等式证明差值函数恒为 0

Lemma 4.1.8 (Gronwall Inequality)

if $u(x)$ is Non-negative Continuous Function on $[x_0, x_1]$, and

$$u(x) \leq C + \int_{x_0}^{x} (a(s)u(s) + K)\,\mathrm{d}s \tag{110}$$

where $C, K \geq 0$, $a(x)$ is Non-negative Continuous Function on $[x_0, x_1]$,

then

$$u(x) \leq (C + K(x - x_0)) e^{\int_{x_0}^{x} a(s)\,\mathrm{d}s} \tag{111}$$

when $C = 0$, then $u(x) \equiv 0$ on $[x_0, x_1]$.

Definition 4.1.9 (Osgood Condition)

For $f(x,y) \in C(G)$ and satisfies

$$|f(x,y_1) - f(x,y_2)| \le F(|y_1 - y_2|) \tag{112}$$

where $F(r) > 0 \in C(0,+\infty)$, and

$$\int_0^{r_1} \frac{\mathrm{d}r}{F(r)} = +\infty \tag{113}$$

then $f(x,y)$ satisfies **Osgood Condition** on G.

Theorem 4.1.10 (Osgood Theorem)

For

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \tag{114}$$

if $f(x,y) \in C(G)$, and $f(x,y)$ satisfies **Osgood Condition** on G then at most only one solution $y = \varphi(x)$ exists on G

4.1.3 Cauchy Theorem of Existence and Uniqueness

Theorem 4.1.11 (Cauchy Theorem)

For

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \\ y(x_0) = y_0 \end{cases} \tag{115}$$

if $f(x,y)$ is **real analytic function** over $D = \{(x,y) |\ |x - x_0| \le a, |y - y_0| \le b\}$

then $\exists!$ one real analytic solution $y = \varphi(x)$ on $U(x_0)$

证明：优级数法(Majorant series method)

Proof. let $(x_0, y_0) = (0,0)$, Then on D,

$$f(x,y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j \tag{116}$$

if $y = \phi(x)$ is real analytic solution, then

$$\phi(x) = \sum_{n=1}^{\infty} b_n x^n \tag{117}$$

and $b_i = p_i(a_{mn})$, where p_i is a polynomial of $a_{mn}, 0 \le m, n \le i-1$.

Consider

$$F(x,y) = \frac{M}{\left(1-\frac{x}{r}\right)\left(1-\frac{y}{r}\right)} \tag{118}$$

where M is const, $r = \min\{a, b\}$, F is **majorant funtion** of f, i.e.

$$F(x,y) = \sum_{i,j=0}^{\infty} A_{ij} x^i y^j \tag{119}$$

where $|a_{ij}| \leq A_{ij}$, we could have serires

$$\sum_{i=0}^{\infty} A_i x^i \tag{120}$$

where $A_i = p_i(A_{mn})$, it's true that this series is **majorant series** of series defined by $\phi(x)$

For

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x,y) \tag{121}$$

we have

$$y - \frac{y^2}{2r} = -rM\ln\left(1-\frac{x}{r}\right) \tag{122}$$

$$y = \Phi(x) = r - r\sqrt{1 + 2M\ln\left(1-\frac{x}{r}\right)} \tag{123}$$

and $\Phi(x)$ is analytic where $|x| < \rho = r\left(1 - e^{-1/2M}\right) \implies \sum_{i=0}^{\infty} b_i x^i \rightrightarrows \phi(x)$

□

4.2 Extension of Solution

Lemma 4.2.1

For $f(x,y) \in C(G)$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \tag{124}$$

- if $\varphi(x)$ is one solution over $[x_0, b)$ and for $\{(x, \varphi(x)) : x \in [x_0, b)\} \subset A \subset G$, where A is **compact set** then $y = \varphi(x)$ could be extended to $[x_0, b]$
- if $\varphi(x) \in [x_0, b]$, $\psi(x) \in [b, c]$ are solution, and $\varphi(b) = \psi(b)$, then

$$y(x) := \begin{cases} \varphi(x), \ x_0 \leq x \leq b \\ \psi(x), \ b < x \leq c \end{cases} \tag{125}$$

is a solution over $[x_0, c]$

♡

证明（1）：紧集$\Longrightarrow$有界$\Longrightarrow$一致连续$\Longrightarrow$边界点极限值存在，且含于紧集$\Longrightarrow$证明积分方程对闭区间都成立

证明（2）：$y(x)$在b处左右连续且左右可微且左右导数连续，证毕

Theorem 4.2.2

For $f(x,y) \in C(G)$, $(x_0, y_0) \in G$ and Γ is integral curve of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \tag{126}$$

through (x_0, y_0), then Γ could be extended to ∂G

Definition 4.2.3 (Saturated Solution)

$\varphi(x)$ is one solution of the problem over I, $\psi(x)$ is another solution over $(a_1, b_1) \supset I$

- if $\forall x \in I$, $\varphi(x) = \psi(x)$ then $\varphi(x)$ is **extendable** and $\psi(x)$ is a **extension** of $\varphi(x)$ over (a_1, b_1)
- if not exists $\psi(x)$, then $\varphi(x)$ is **saturated**

Theorem 4.2.4

if $f(x,y) \in C(G)$, $G \subset \mathbb{R}^2$ and satisfies Lipschitz condition for y, then for $\forall(x,y) \in G$, $\exists!$ saturated solution through it.

证明：反证法$\Longrightarrow$假设在某个点上两饱和解分开$\Longrightarrow$由条件得该点邻域连续且满足 Lipschitz 条件$\Longrightarrow$由 Picard 定理得该点邻域解唯一$\Longrightarrow$矛盾

4.3 Global Existence of Solutions

Theorem 4.3.1

For $f(x,y) \in C(R)$ where $R = \{(x,y) | a < x < b, |y| < +\infty\}$, and satisfies **Global Lipschitz Condition** for y, then for

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \tag{127}$$

then every **saturated solution** is confined to (a, b)

Theorem 4.3.2 (1st Comparison Theorem)

For $f(x,y), F(x,y) \in C(G)$, $y = \phi(x)$ and $y = \Phi(x)$ are solutions of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \tag{128}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x,y) \tag{129}$$

on $[x_0, b)$, $y(x_0) = y_0$ and if $\forall(x,y) \in G$

$$f(x,y) < F(x,y) \tag{130}$$

then for $x \in (x_0, b)$, $\phi(x) < \Phi(x)$

Definition 4.3.3

on $[x_0, b)$, if

$$\frac{\mathrm{d}v}{\mathrm{d}x} < f(x,y) = \frac{\mathrm{d}y}{\mathrm{d}x} < \frac{\mathrm{d}w}{\mathrm{d}x}, \\ v(x_0) \le y_0 = y(x_0) \le w(x_0) \tag{131}$$

then $v(x)$, $y(x)$, $w(x)$ are called **Lower Solution(Right)**, **Solution(Right)**, **Upper Solution(Right)** respectively.

Theorem 4.3.4

for $f(x,y) \in C(I)$, if $v(x), w(x)$ are **Lower Solution(Right)**, **Upper Solution(Right)** on $[x_0, b)$ of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y),\ y(x_0) = y_0 \tag{132}$$

then on (x_0, b)

$$v(x) < y(x) < w(x) \tag{133}$$

Theorem 4.3.5

if $f(x,y) \in C(R)$ where $R = \{(x,y)|x_0 \le x \le x_0 + a, |y - y_0| \le b\}$, then for

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)\ ,\quad y(x_0) = y_0 \tag{134}$$

then there exists **lowest solution** and **uppest solution** $y = V(x)$ and $y = W(x)$ on $[x_0, x_0 + h]$, where

$$h < \alpha = \min\left(a, \frac{b}{M}\right), M = \sup_{(x,y)\in R} |f(x,y)| \tag{135}$$

i.e. for any solution $y = \varphi(x)$ on $[x_0, x_0 + h]$,

$$V(x) \le \varphi(x) \le W(x) \tag{136}$$

Theorem 4.3.6 (2nd Comparison Theorem)

if $f(x,y) \in C(G)$, where $G = \{(x,y)|\ x_0 \le x < b, |y| < \infty\}$, for $(x_0, y_0) \in G$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y),\ y(x_0) = y_0 \tag{137}$$

the maximum range is $[x_0, \beta_1)$

- if upper solution $\Phi(x)$ and lower solution $\varphi(x)$ have public range $[x_0, \beta)$ then $\beta_1 \ge \beta$
- if exists upper(or lower) solution $\Phi(x)$ $(\varphi(x))$, and its maximum range is $[x_0, \beta)$, where $x \to \beta^-$, $\Phi(x) \to -\infty$ (or $\varphi(x) \to +\infty$) , then $\beta_1 \le \beta$

4.4 Continuous Dependence of Solutions

Theorem 4.4.1 (Continuous Dependence of Evolution Function)

$F(x,y), f(x,y) \in C(G)$ and satisfies

$$|F(x,y) - f(x,y)| \leq \varepsilon \tag{138}$$

$f(x,y)$ satisfies **Lipschitz condition** for y (Lip const= L) For

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x,y), \ \dfrac{\mathrm{d}\varphi}{\mathrm{d}x} = F(x,y) \\ y(x_0) = \varphi(x_0) = y_0 \end{cases} \tag{139}$$

on $|x - x_0| \leq \alpha$, we have

$$|y(x) - \varphi(x)| \leq \frac{\varepsilon}{L}(e^{L\alpha} - 1) \tag{140}$$

Theorem 4.4.2 (Continuous Dependence of Initial Value)

$G \subset \mathbb{R}$ is connected, $f(x,y) \in C(G)$ and satisfies **Local Lipschitz Condition** for y if

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \\ y(x_0) = x_0 \end{cases} \tag{141}$$

has unique solution $y = \varphi(x;\, x_0, y_0)$ on $I = [a,b] \subset G$, then over I

$$\lim_{(\xi,\eta)\to(x_0,y_0)} \varphi(x;\, \xi, \eta) \rightrightarrows \varphi(x;\, x_0, y_0) \tag{142}$$

Theorem 4.4.3 (Continuous Dependence of Initial Value + Parameter)

$f(x,y,\lambda) \in C(G_\lambda)$ and satisfies **Local Lipschitz Condition** for (y,λ) if

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x,y,\lambda) \\ y(x_0) = x_0 \\ \lambda = \lambda_0 \end{cases} \tag{143}$$

has solution $y = \varphi(x;\, x_0, y_0, \lambda_0)$ on $I = [a,b] \subset G_\lambda$, then over I

$$\lim_{(\xi,\eta,\lambda)\to(x_0,y_0,\lambda_0)} \varphi(x;\, \xi, \eta, \lambda) \rightrightarrows \varphi(x;\, x_0, y_0, \lambda_0) \tag{144}$$

Theorem 4.4.4 (Differentiable Properties of Solution + Parameter)

$f(x,y,\lambda) \in C(G_\lambda)$ and satisfies **Local Lipschitz Condition** for (y,λ)

$f_y(x,y,\lambda),\ f_\lambda(x,y,\lambda) \in C(G_\lambda)$ then $y = \varphi(x;\, x_0, y_0, \lambda)$is **Continueous and Differentiabe**

$$u = \frac{\partial \varphi}{\partial x_0}, v = \frac{\partial \varphi}{\partial y_0}, w = \frac{\partial \varphi}{\partial \lambda} \tag{145}$$

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}x} = f_y(x, \varphi(x;\, x_0, y_0, \lambda))u \\ u(x_0) = -f(x_0, y_0, \lambda) \end{cases} \tag{146}$$

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}x} = f_y(x, \varphi(x;\, x_0, y_0, \lambda))v \\ v(x_0) = 1 \end{cases} \tag{147}$$

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}x} = f_y(x, \varphi(x;\, x_0, y_0, \lambda))u + f_\lambda(x, \varphi(x;\, x_0, y_0, \lambda)) \\ w(x_0) = 0 \end{cases} \tag{148}$$

4.5 ODEs and Higher Order ODEs

- [x] **Uniqueness and Existence**
- [x] **Extension**
- [ ] **Comparison**
- [x] **Continueous Dependence**

4.6 First Integral

Definition 4.6.1

For non-trivial $\psi(t, \boldsymbol{X}) \in C^1(\mathbb{R}^{n+1})$ and if solutions of

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{F}(t, \boldsymbol{X}) \tag{149}$$

could be expressed in

$$\psi(t, \boldsymbol{X}) = C \tag{150}$$

then it's a **First Integral**

if $\psi_j(t, \boldsymbol{X}) = C_j (1 \le j \le n)$ satisfies

$$\frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(x_1, \ldots, x_n)} \neq 0 \tag{151}$$

then they are **Independent**

> the number of First Integral won't be larger than n at most

5 Stability Theory

5.1 Lyapunov Stability

Definition 5.1.1 ((Lyapunov) Stable)

For

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{t}, \boldsymbol{x}) \\ t \in (\beta, +\infty) \end{cases}, \ \beta \in \overline{\mathbb{R}} \tag{152}$$

$\boldsymbol{f}(t, \boldsymbol{x})$ satisfies **Local Lipschitz Condition** for $\boldsymbol{x}$ and has a solution $\boldsymbol{x} = \boldsymbol{\varphi}(t)$

For $\forall \varepsilon > 0$, $\forall t_0 > \beta$, $\exists \delta(t_0, \varepsilon) > 0$ s.t. $\|\boldsymbol{x}_0 - \boldsymbol{\varphi}(\boldsymbol{t_0})\| < \delta$, and if $\forall t > t_0$,

$$\|\boldsymbol{x}(t;\, t_0, \boldsymbol{x}_0) - \boldsymbol{\varphi}(t)\| < \varepsilon \tag{153}$$

then $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ is **Lyapunov Stable** if $\delta = \delta(\varepsilon)$ then **Global Stable**

Definition 5.1.2 ((Lyapunov) Attractive)

For $\forall t_0 > \beta$, $\exists \delta_1(t_0)$ s.t. $\|\boldsymbol{x}_0 - \boldsymbol{\varphi}(\boldsymbol{t_0})\| < \delta_1$, and if $\forall t > t_0$,

$$\lim_{t \to +\infty} \|\boldsymbol{x}(t;\, t_0, \boldsymbol{x}_0) - \boldsymbol{\varphi}(t)\| = 0 \tag{154}$$

then $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ is **Lyapunov Attractive** if $\delta_1(t_0) \equiv \delta_1$ then **Global Attractive**

Note

Asymptotically Stable = Stable + Attractive

Global Asymptotically Stable = Global Stable + Global Attractive

5.2 Stability Analysis

Definition 5.2.1

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{R}(t, \boldsymbol{x}) \\ \boldsymbol{A}(t) = \left[\dfrac{\partial \boldsymbol{f}(t, \boldsymbol{x})}{\partial \boldsymbol{x}}\right]_{x=0} = \nabla_{\boldsymbol{x}} \boldsymbol{f}\Big|_{\boldsymbol{x}=0} \\ \boldsymbol{R}(t, \boldsymbol{x}) = O(\boldsymbol{x}) \end{cases} \tag{155}$$

if we have

$$\lim_{\|\boldsymbol{x}\| \to 0} \frac{\|\boldsymbol{R}(t, \boldsymbol{x})\|}{\|\boldsymbol{x}\|} = 0 \tag{156}$$

then

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}(t)\boldsymbol{x} \tag{157}$$

is a **Linear Approximation Equation**

Theorem 5.2.2

For

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}(t)\boldsymbol{x} \tag{158}$$

we have $\boldsymbol{\Phi}(t)$ satisfies $\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{C}$

- **Zero Solution is Stable** $\iff \exists K = K(t_0) > 0$ s.t.

$$\|\boldsymbol{\Phi}(t)\| \leq K,\ t \geq t_0 \tag{159}$$

- **Zero Solution is Stable** $\iff \exists K = K(\beta) > 0$ s.t.

$$\left\|\boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(s)\right\| \leq K,\ t > s > \beta \tag{160}$$

- **Zero Solution is Asymptotically Stable** $\iff$

$$\lim_{t \to +\infty} \|\boldsymbol{\Phi}(t)\| = 0 \tag{161}$$

- **Zero Solution is Global Asymptotically Stable** $\iff \exists M = M(\beta) > 0,\ \alpha = \alpha(\beta) > 0$ s.t.

$$\left\|\boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(s)\right\| \leq Me^{-\alpha(t-s)},\ t > s > \beta \tag{162}$$

Theorem 5.2.3

For

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x} \tag{163}$$

λ_j are Eigenvalues of $\boldsymbol{A}$

- **Zero Solution is (Global) Asymptotically Stable** $\iff \forall j = 1, ..., n$

$$\mathrm{Re}(\lambda_j) < 0 \tag{164}$$

- **Zero Solution is (Global) Stable** $\iff \forall j, k = 1, ..., n$

$$\mathrm{Re}(\lambda_j) < 0,\ \mathrm{Re}(\lambda_k) = 0,\ \dim(\boldsymbol{J}_k) = 1 \tag{165}$$

- **Zero Solution is Unstable** $\iff \exists j, k = 1, ..., n$

$$\mathrm{Re}(\lambda_j) > 0 \text{ or } \mathrm{Re}(\lambda_k) = 0,\ \dim(\boldsymbol{J}_k) \geq 2 \tag{166}$$

Theorem 5.2.4

For

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{R}(t, \boldsymbol{x}) \tag{167}$$

if **Zero solution** of

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}(t)\boldsymbol{x} \tag{168}$$

is **Global Stable** and $\exists \delta > 0$, and $b(t)$ satisfies

$$\int_{\beta}^{+\infty} b(t)\,\mathrm{d}t < +\infty \tag{169}$$

then s.t.

$$\|\boldsymbol{R}(t, \boldsymbol{x})\| \leq b(t)\|\boldsymbol{x}\|,\ t > \beta,\ \|\boldsymbol{x}\| \leq \delta \tag{170}$$

then **Zero solution** of (167) is **Global Stable**

♡

Theorem 5.2.5

if **Zero solution** of

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}(t)\boldsymbol{x} \tag{171}$$

is **Global Asymptotically Stable** and $\exists \delta > 0$, and $b(t)$ satisfies

$$\int_{t_0}^{t} b(s)\,\mathrm{d}s < \gamma(\beta)(t - t_0) + \tau(\beta), \quad t > t_0 > \beta \tag{172}$$

then s.t.

$$\|\boldsymbol{R}(t, \boldsymbol{x})\| \leq b(t)\|\boldsymbol{x}\|,\ t > \beta,\ \|\boldsymbol{x}\| \leq \delta \tag{173}$$

then if $\exists r > 0$ when $\gamma < r$, **Zero solution** of (167) is **Global Stable**

♡

> 类似 Lipshitz 条件

Theorem 5.2.6

For

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{R}(t, \boldsymbol{x}) \tag{174}$$

if $\boldsymbol{R}(t, \boldsymbol{x})$ satisfies

$$\lim_{\|\boldsymbol{x}\| \to 0} \frac{\|\boldsymbol{R}(t, \boldsymbol{x})\|}{\|\boldsymbol{x}\|} = 0, \quad \forall t > \beta \tag{175}$$

♡

- **Zero solution** of (174) is **Global Asymptotically Stable** $\iff \forall\, \mathrm{Re}(\lambda_A) < 0$
- **Zero solution** of (174) is **Unstable** $\iff \exists\, \mathrm{Re}(\lambda_A) > 0$

Theorem 5.2.7 (Routh-Hurwitz Laws)

if

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n \tag{176}$$

Then for

$$\Delta_n = \begin{pmatrix} a_1 & a_0 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_n \end{pmatrix} \tag{177}$$

All Real Part of Roots of $P_n(\lambda)$ are negative $\iff \forall$ **Leading Principle Minor(顺序主子式)** of Δ_n are positive

5.3 Autonomous System and Lyapunov 2nd Method

Definition 5.3.1 (Lyapunov Function)

$V(\boldsymbol{x}) \in C^1(I)$, $I = \{\|\boldsymbol{x}\| < h\}$

Based on different some condition below

- **I. Postive-Definite:** $V(0) = 0; V(\boldsymbol{x}) > 0,\ \boldsymbol{x} \neq 0$
- **II. Negative-Definite Derivate:** $\dfrac{DV}{Dt} = \nabla_{\boldsymbol{x}} V \cdot \boldsymbol{f} < 0,\ \boldsymbol{x} \neq 0$
- **III. Semi-Negative-Definite Derivate:** $\dfrac{DV}{Dt} = \nabla_{\boldsymbol{x}} V \cdot \boldsymbol{f} \leq 0$
- **IV. Positive-Definite Derivate:** $\dfrac{DV}{Dt} = \nabla_{\boldsymbol{x}} V \cdot \boldsymbol{f} > 0,\ \boldsymbol{x} \neq 0$

Theorem 5.3.2

For **Autonomous System**

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}) \tag{178}$$

- **I+II** $\Longrightarrow$ **Zero solution** of (178) is **Asymptotically Stable**
- **I+III** $\Longrightarrow$ **Zero solution** of (178) is **Stable**
- **I+IV** $\Longrightarrow$ **Zero solution** of (178) is **Unstable**